Sol^n of Δ

Δ : Area of triangle

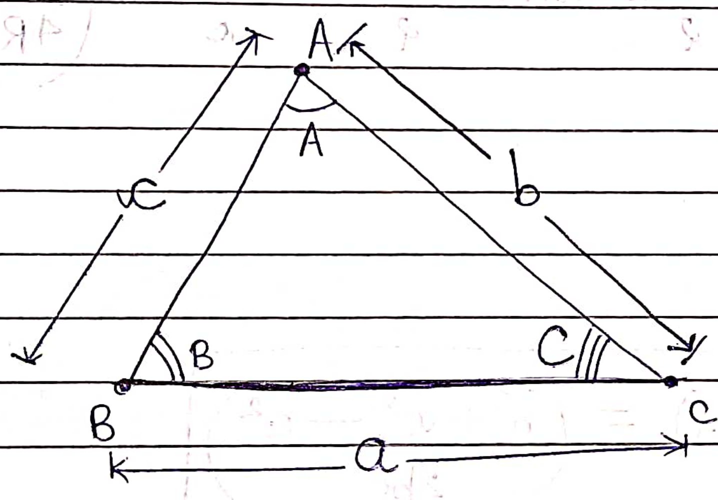
s : Semiperimeter $\left\{ s = \left(\frac{a+b+c}{2}\right) \right\}$

r : Inradius

R : Circumradius

r_A, r_B, r_C : Ex-radii

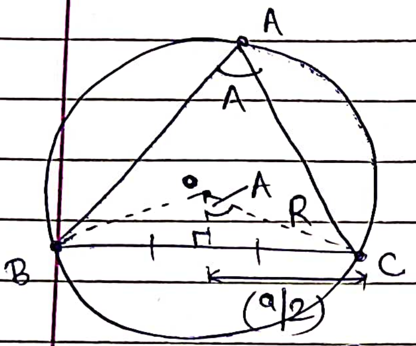
I_A, I_B, I_C : Ex-centres



Sine Rule

$$\boxed{\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} = \left(\frac{1}{2R}\right) = \left(\frac{2\Delta}{abc}\right)}$$

Derivation:



$$\sin(A) = \frac{(a/2)}{R}$$

$$\Rightarrow \frac{\sin(A)}{a} = \frac{1}{2R}$$

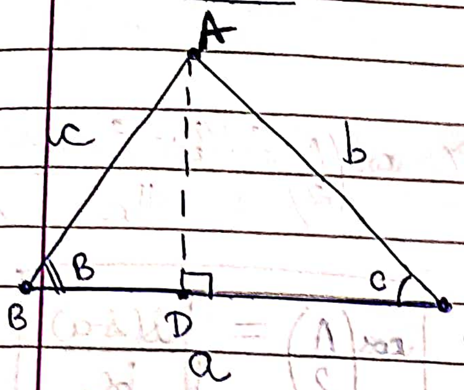
$$\Delta = \frac{1}{2} ab \sin(C) = \frac{1}{2} abc \frac{\sin(C)}{c} = \frac{abc}{4R}$$

Cosine Rule

$$\boxed{\cos(A) = \left(\frac{b^2 + c^2 - a^2}{2bc}\right)}$$

Similarly for B & C.

Derivation:



$$b^2 = (DC)^2 + (AD)^2$$
$$c^2 = (BD)^2 + (AD)^2$$
$$a^2 = (BD)^2 + (DC)^2 + 2\,BD \cdot DC$$
$$\Rightarrow (b^2 + c^2 - a^2) = (2)\left[(AD)^2 - BD \cdot DC\right]$$

Now, $\cos(A) = -\cos(B+C)$
$$= -\cos(B)\cos(C) + \sin(B)\sin(C)$$
$$= -\frac{BD}{c}\cdot\frac{DC}{b} + \frac{AD}{c}\cdot\frac{AD}{b} = \left[\frac{(AD)^2 - BD \cdot DC}{bc}\right]$$

Hence, $\cos(A) = \left(\frac{b^2 + c^2 - a^2}{2bc}\right)$

Half Angle formulae

$$\boxed{\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}}$$

$$\boxed{\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}}$$

$$\boxed{\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}$$

Similarly for B & C.

Derivation :

$$\cos(A) = \boxed{2\cos^2\left(\frac{A}{2}\right) - 1} = \left(\frac{b^2+c^2-a^2}{2bc}\right) \Rightarrow \cos^2\left(\frac{A}{2}\right) = \left(\frac{(b+c)^2-a^2}{4bc}\right)$$

$$\Rightarrow \cos^2\left(\frac{A}{2}\right) = \left(\frac{\left(\frac{b+c+a}{2}\right)\left(\frac{b+c-a}{2}\right)}{bc}\right) \Rightarrow \boxed{\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}}$$

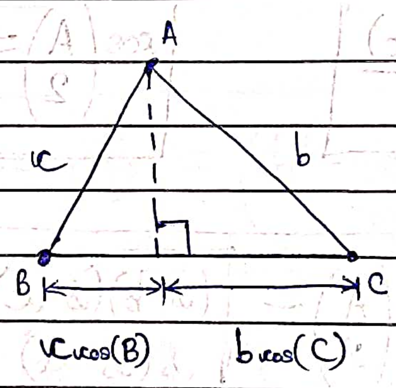
Now, $\sin^2\left(\frac{A}{2}\right) = \boxed{1 - \cos^2\left(\frac{A}{2}\right)} = \left(\frac{a^2-(b-c)^2}{4bc}\right)$

$$= \frac{\left(\frac{a+b-c}{2}\right)\left(\frac{a+c-b}{2}\right)}{bc} \Rightarrow \boxed{\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}}$$

Projection formulae

$$\boxed{a = b\cos(C) + c\cos(B)}$$

Derivation:

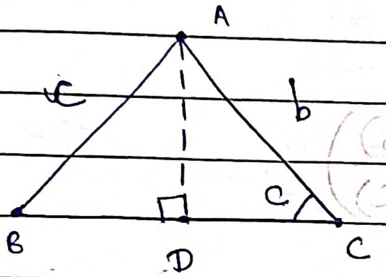


Similarly for b & c

Area of Δ

$$\boxed{\begin{aligned}\Delta &= \frac{1}{2} bc \sin(A) = \frac{1}{2} ca \sin(B) = \frac{1}{2} ab \sin(C) \\ &= \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = rs\end{aligned}}$$

Derivation :



$$\Delta = \frac{1}{2} \cdot BC \cdot AD = \frac{1}{2} \cdot a \cdot b \sin(C)$$

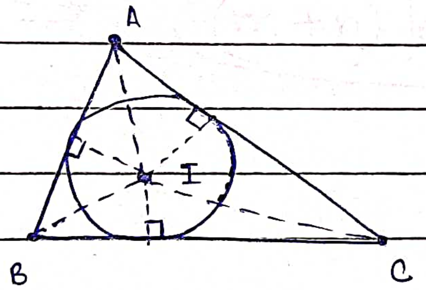
$$= \frac{1}{2} ab \cdot 2 \sin\left(\frac{C}{2}\right) \cos\left(\frac{C}{2}\right)$$

$$= ab \cdot \sqrt{\frac{(s-a)(s-b)}{ab}} \sqrt{\frac{s(s-c)}{ab}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

Also, $\sin(A) = \dfrac{a}{2R} \Rightarrow \Delta = \dfrac{1}{2} \cdot bc \sin(A) = \dfrac{abc}{4R}$

Also,



$$\Delta = [\Delta AIB] + [\Delta BIC] + [\Delta CIA]$$

$$= \frac{1}{2} \cdot r \cdot c + \frac{1}{2} \cdot r \cdot a + \frac{1}{2} \cdot r \cdot b$$

$$= r\left(\frac{a+b+c}{2}\right) = rs$$

Napier's Analogy

$$\boxed{\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)}$$

Similarly for B & C.

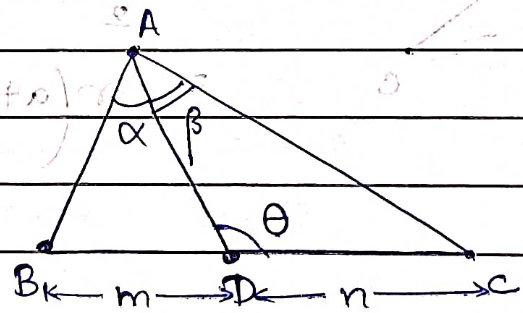
Derivation :

$$\frac{\sin(B)}{b} = \frac{\sin(C)}{c} \Rightarrow \left(\frac{b}{c}\right) = \left(\frac{\sin(B)}{\sin(C)}\right)$$

$$\Rightarrow \left(\frac{b+c}{b-c}\right) = \left(\frac{s_B + s_C}{s_B - s_C}\right) = \left(\frac{\cancel{2}\, s_{\left(\frac{B+C}{2}\right)}\, c_{\left(\frac{B-C}{2}\right)}}{\cancel{2}\, s_{\left(\frac{B-C}{2}\right)}\, c_{\left(\frac{B+C}{2}\right)}}\right)$$

$$= \frac{t_{\left(\frac{B+C}{2}\right)}}{t_{\left(\frac{B-C}{2}\right)}} = \frac{T_{A/2}}{t_{\left(\frac{B-C}{2}\right)}} \Rightarrow t_{\left(\frac{B-C}{2}\right)} = \left(\frac{b-c}{b+c}\right) T_{(A/2)}$$

m-n Theorem



$$\boxed{(m+n)T_\theta = (mT_\alpha - nT_\beta)}$$

$$\boxed{(m+n)T_\theta = (nT_B - mT_C)}$$

Derivation :

$$\frac{m}{\sin(\alpha)} = \frac{AD}{\sin(B)} \quad \& \quad \frac{n}{\sin(\beta)} = \frac{AD}{\sin(C)}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{\sin(C)\sin(\alpha)}{\sin(B)\sin(\beta)} = \frac{\sin(\alpha)\sin(\theta+\beta)}{\sin(\beta)\sin(\theta-\alpha)}$$

$$= \frac{s_\alpha(s_\theta c_\beta + c_\theta s_\beta)}{s_\beta(s_\theta c_\alpha - c_\theta s_\alpha)} = \left(\frac{T_\beta + T_\theta}{T_\alpha - T_\theta}\right)$$

$$\Rightarrow \quad (mT_\alpha - nT_\beta) = (m+n)T_\theta$$

Also,

$$\frac{m}{n} = \frac{\sin(C)\sin(\alpha)}{\sin(B)\sin(\beta)} = \frac{\sin(C)\sin(\theta-B)}{\sin(B)\sin(\theta+C)}$$

$$= \frac{s_C(s_\theta c_B - c_\theta s_B)}{s_B(s_\theta c_C + c_\theta s_C)} = \left(\frac{T_B - T_\theta}{T_C + T_\theta}\right)$$

$$\Rightarrow \quad (m+n)T_\theta = (nT_B - mT_C)$$

Circles connected with Triangle

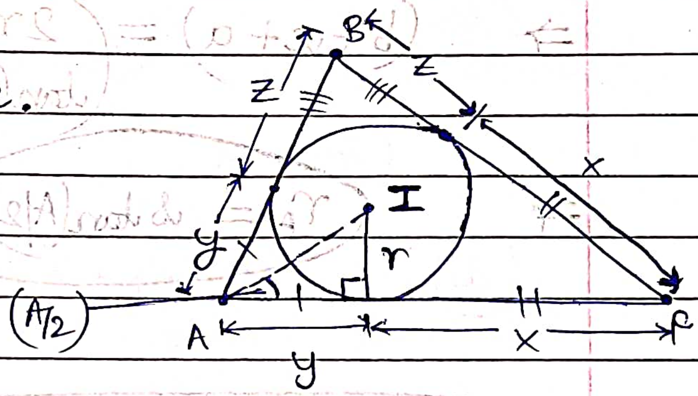
1) $\boxed{\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)} = 2R}$

2) $\boxed{R = \dfrac{abc}{4\Delta}}$

3) $\boxed{r = \dfrac{\Delta}{s} = (s-a)\tan\left(\dfrac{A}{2}\right)}$

4) $\boxed{r = 4R \sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right)}$

Proof: 3) Construct incircle.



$x+y+z = s$

$x+z = a$

$\Rightarrow y = (s-a)$

$\tan(A/2) = r/y = r/(s-a) \Rightarrow r = (s-a)\tan(A/2)$

5) $\boxed{r_A = \dfrac{\Delta}{(s-a)} = s\tan\left(\dfrac{A}{2}\right) = 4R\ \cancel{\sin}\cos\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\cancel{\cos}\sin\left(\dfrac{C}{2}\right)}$

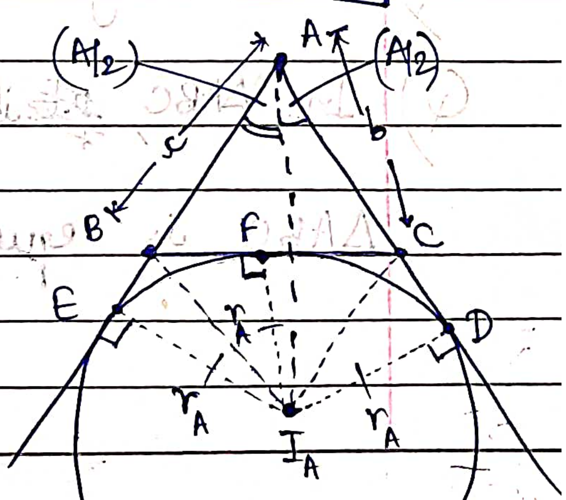
Proof: Draw excircle.

$\Delta = [AI_AC] + [AI_AB] - [BI_AC]$

$= br_A/2 + cr_A/2 - ar_A/2$

$= (b+c-a)r_A/2$

$\Rightarrow \cancel{(s-a)r_A} \quad r_A = \dfrac{\Delta}{(s-a)}$



$\tan(A/2) = \left(\frac{r_A}{c+BE}\right)$

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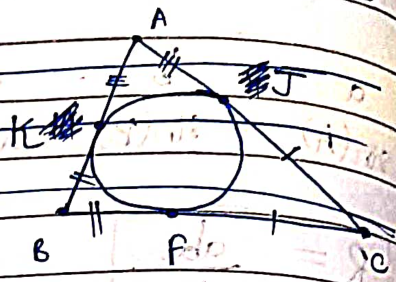
Obviously, $\tan(A/2) = \left(\frac{r_A}{b+CD}\right)$



~~Now, $FC = CD = CJ$~~

~~Now, $CJ + AK + BF = s$~~

~~& $AK + BF = c$~~

~~$\Rightarrow CJ = (s-c) = CD$~~

$\Rightarrow \left(b+c+BE+CD\right) = \left(\frac{2r_A}{\tan(A/2)}\right)$

$\Rightarrow \left(b+c+a\right) = \left(\frac{2r_A}{\tan(A/2)}\right)$

$\Rightarrow$ I $r_A = s\tan(A/2)$

6) $\boxed{\left(r_A r_B + r_B r_C + r_C r_A\right) = s^2 = \left(\frac{r_A r_B r_C}{r}\right)}$

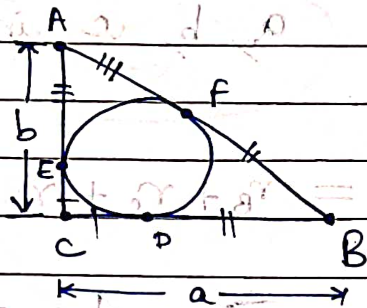
Q) In a right angled ΔABC, p.t. $r + 2R = s$.

Q) In ΔABC, ~~p.t.~~ if $\left(\frac{\sum a r_A}{\sum a s_B}\right) = \left(\frac{\sum a}{9R}\right)$ p.t. ΔABC is equi.

Q) If r_A, r_B, r_C in H.P.; p.t. a, b, c in A.P.

Q) In a Δ, if $r_A = r_B + r_C + r$; p.t. it is right angled

A) $\Delta = \frac{1}{2} ab$



$\Rightarrow R = \frac{abc}{4\Delta} = \frac{c}{2}$

~~Not~~ Now, $r + 2R = r + c$

$= \frac{1}{2}(2r + 2c)$

$= \frac{1}{2}(EC + CD + Af + fB + Af + fB)$

$= \frac{1}{2}(EC + CD + DB + Bf + fA + AE) = s$

A) $\left(\frac{\sum a c_A}{\sum a s_B}\right) = \frac{\sum(c_A \cdot s_A \cdot 2R)}{\sum(s_B \cdot s_A \cdot 2R)} = \frac{\sum(s_{2A})}{\sum(s_A s_B)} = \frac{4 s_A s_B s_C}{2\sum(s_A s_B)}$

$= \frac{4}{2 \cdot \sum(1/s_A)} \underset{(AM \geq HM)}{\leq} \frac{4}{2} \cdot \frac{\sum s_A}{9} = \frac{4}{2 \cdot 9} \cdot \sum\left(\frac{a}{2R}\right) = \left(\frac{\sum a}{9R}\right)$

We are given equality case $\Rightarrow s_A = s_B = s_C$

$\Rightarrow \boxed{\text{Equi } \Delta}$

A) r_A, r_B, r_C in H.P. $\Rightarrow \frac{\Delta}{(s-a)}, \frac{\Delta}{(s-b)}, \frac{\Delta}{(s-c)}$ in H.P.

$\Rightarrow (s-a), (s-b), (s-c)$ in A.P.

$\Rightarrow a, b, c$ in A.P.

A) $r_A = r_B + r_C + r \Rightarrow \left(\frac{r_A}{r}\right) = \left(\frac{r_B}{r}\right) + \left(\frac{r_C}{r}\right) + 1$

$\Rightarrow T_{A/2} = T_{B/2} + T_{C/2} + 1$

$\Rightarrow ct_{\left(\frac{B+C}{2}\right)} = \left(T_{\frac{B}{2}} + T_{\frac{C}{2}} + 1\right) = \left(\frac{T_{B/2} + T_{C/2}}{T_{B/2} \cdot T_{C/2} - 1}\right)$

$\Rightarrow$

Q) In ΔABC, p.t. $\sum\left(T_{A/2}\right) = \prod\left(T_{A/2}\right)$

Q) Let ΔABC with incentre I & inradius 'r'. Let D, E, F be feet of ⊥rs from I onto sides ~~BC, AB~~ BC, AC, AB resp. If r_1, r_2, r_3 are radii of circles in quads. AFIE, BDIF, CEID resp.

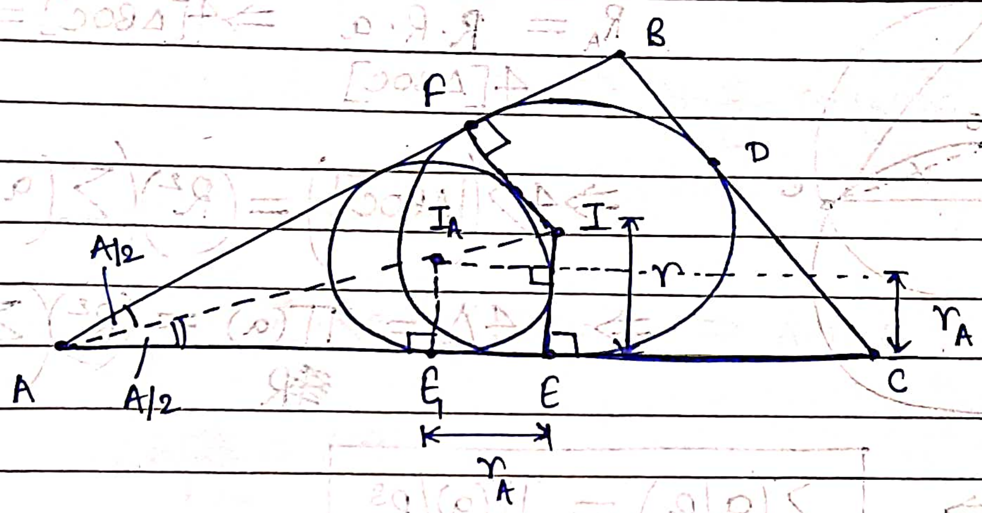
P.t. $\sum\left(\frac{r_A}{r - r_A}\right) = \left(\frac{r_A r_B r_C}{\prod(r - r_A)}\right)$

A) $\sum\left(T_{A/2}\right) = \prod\left(T_{A/2}\right) \Rightarrow \sum\left(t_{B/2}\, t_{C/2}\right) = 1$ ✓

Now, $\left(\dfrac{t_{B/2}+t_{C/2}}{1-t_{B/2}\,t_{C/2}}\right) = T_{\frac{A}{2}} \Rightarrow \sum\left(t_{\frac{A}{2}} t_{\frac{B}{2}} + t_{\frac{C}{2}} t_{\frac{A}{2}}\right) = \sum\left(1 - t_{\frac{B}{2}} t_{\frac{C}{2}}\right)$

$\Rightarrow \sum\left(t_{\frac{B}{2}} t_{\frac{C}{2}}\right) = 1$

A)



$$\tan(A/2) = \left(\frac{r}{AE}\right) = \left(\frac{r_A}{AE_1}\right) = \left(\frac{r-r_A}{EE_1}\right) = \left(\frac{r-r_A}{r_A}\right)$$

$\Rightarrow T_{A/2} = \left(\dfrac{r_A}{r-r_A}\right)$ Using above identity.

$$\Rightarrow \sum\left(\frac{r_A}{r-r_A}\right) = \prod\left(\frac{r_A}{r-r_A}\right) ✓$$

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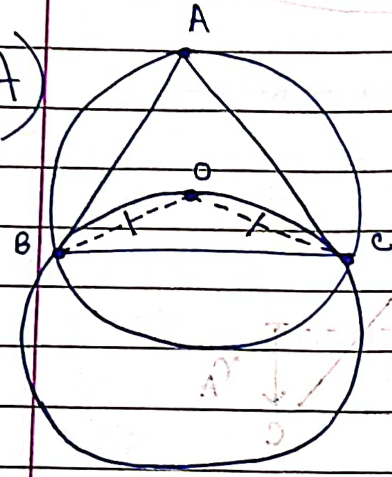
Q) If 'O' be the circumcentre of acute angled ΔABC and R_A, R_B, R_C are resp. the radii of the circumcircle of $\Delta OBC, \Delta OCA, \Delta OAB$. Then p.t. $\sum (a/R_A) = \prod(a)/R^3$

A) 

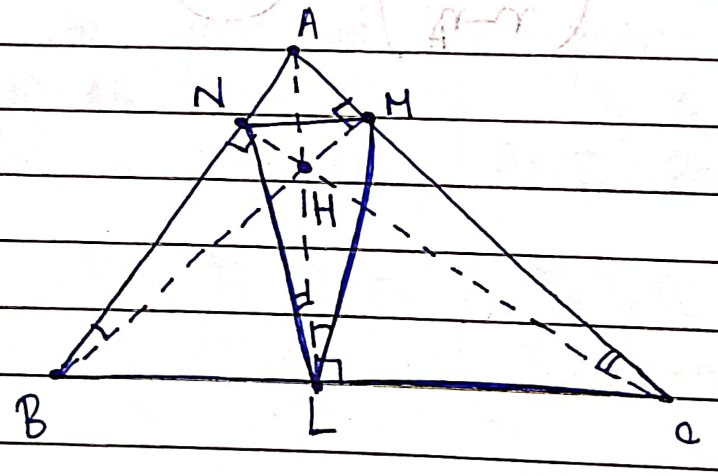
$$R_A = \frac{R \cdot R \cdot a}{4[\Delta BOC]} \Rightarrow 4[\Delta BOC] = (R^2)\left(\frac{a}{R_A}\right)$$

$$\Rightarrow 4\sum([\Delta BOC]) = (R^2)\left(\sum(a/R_A)\right)$$

$$\Rightarrow 4\Delta = \frac{\prod(a)}{R} = (R^2)\left(\sum(a/R_A)\right)$$

$$\Rightarrow \boxed{\sum(a/R_A) = \prod(a)/R^3}$$

Orthocentre & Pedal Δ



ΔLMN is Pedal Δ of ΔABC.

H is Orthocentre of ΔABC.

Imp. Pts –

1) H is incentre of ΔLMN.

Proof: $\angle AMB = \angle ALB = 90° \Rightarrow$ AMLB concyclic

$\Rightarrow \angle ALM = \angle ABM$

Similarly, $\angle ANC = \angle ALC = 90° \Rightarrow$ ANLC concyclic

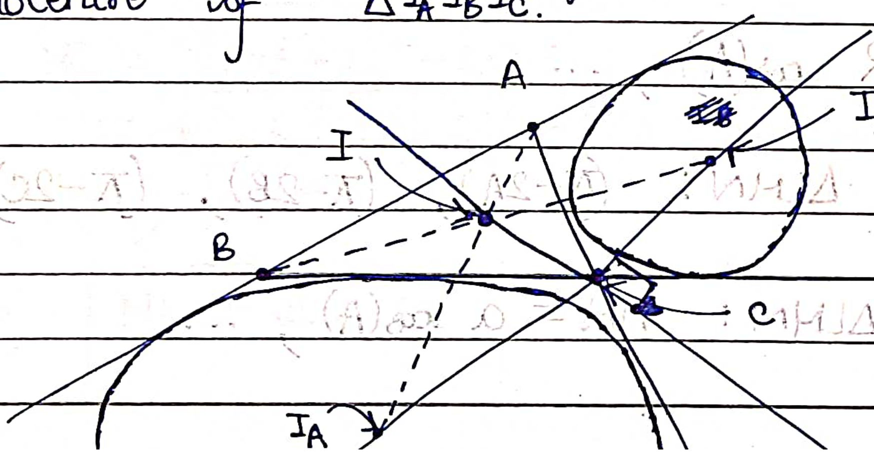
$\Rightarrow \angle ALN = \angle ACN$

Now, $\angle ABM = \angle ACN = (90° - \angle A) \Rightarrow \angle ALM = \angle ALN$

$\Rightarrow$ AL bisects $\angle NLM \Rightarrow$ H is incentre of ΔLMN.

2) If I_A, I_B, I_C be centres of escribed $\odot$s of ΔABC & I be centre of incircle the ΔABC is pedal Δ of $\Delta I_A I_B I_C$ & I is orthocentre of $\Delta I_A I_B I_C$.

Proof:



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Both CI_A & CI_B bisect external angles at C $\Rightarrow$ I_A, C, I_B are collinear.

We know, I_C, I, C are collinear.

$\Rightarrow$ $I_C I$ and $I_A I_B$ are int. & ext. angle bisector of $\angle C$ resp.

$\Rightarrow$ $\angle ICI_A = \angle ICI_B = 90$ & IC passes thru I_C

$\Rightarrow$ C is foot of $\perp$ from I_C to $I_A I_B$.

$\Rightarrow$ ΔABC is pedal Δ of $\Delta I_A I_B I_C$ & I is orthocentre of $\Delta I_A I_B I_C$

3) Circum~~circle~~centre of pedal Δ of a Δ divides line joining ~~circumcentre~~ circumcentre & orthocentre of the Δ in 1:1

(Proof using Nine Pt. ⊙)

4) ~~PL~~ HL $= 2R\cos(B)\cos(C)$, ...

5) ~~AP~~ AH $= 2R\cos(A)$, ...

6) Angles of ΔLMN: $(\pi-2A)$, $(\pi-2B)$, $(\pi-2C)$

7) Sides of ΔLMN: $MN = a\cos(A)$, ...

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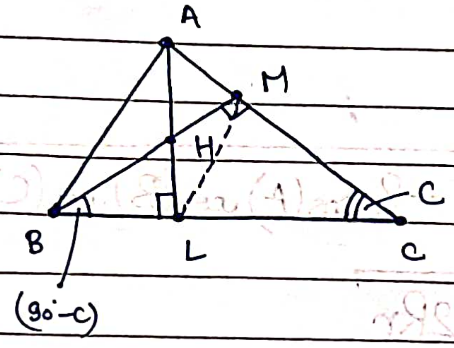
Proof:



$\tan(90°-C) = \dfrac{HL}{BL}$

$BL = c\cos(B)$

$\Rightarrow \cot(C) = \dfrac{HL}{c\cos(B)}$

$\Rightarrow HL = \dfrac{c}{\sin(C)}\cos(B)\cos(C)$

$\Rightarrow HL = 2R\cos(B)\cos(C)$

Now, $AH = (AL - HL) = b\sin(C) - 2R\cos(B)\cos(C)$

$= 2R\, s_B s_C - 2R\, c_B c_C$

$= -2R\, c_{(B+C)} = -2R\, c_{(\pi-A)}$

$\Rightarrow AH = 2R\cos(A)$

Earlier we found AMBL concyclic (as $\angle AMB = \angle ALB = 90°$)

$\Rightarrow \angle ALM = \angle ABM = (90° - A)$

Now, $\angle NLM = 2\angle ALM = (\pi - 2A)$

Now, $ML = \sqrt{MC^2 + LC^2 - 2\cdot MC\cdot LC\cdot\cos(C)}$

$= \sqrt{(a c_C)^2 + (b c_C)^2 - 2(a c_C)(b c_C) c_C}$

$= (c_C)\sqrt{a^2+b^2-2ab\, c_C}$

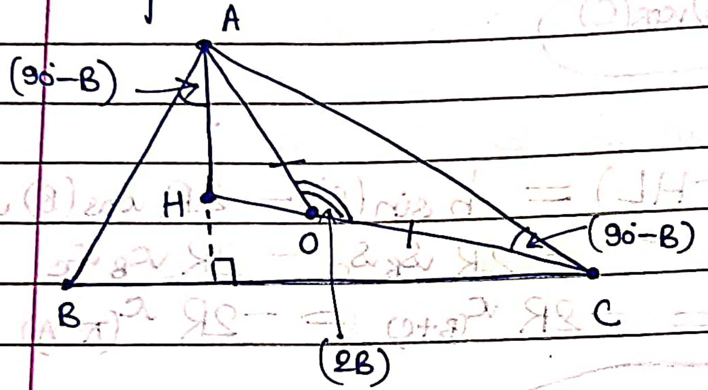
$\Rightarrow \boxed{ML = c\cos(C)}$

Dist. b/w Imp. Pts.

1) H & O : $R\sqrt{1-8\cos(A)\cos(B)\cos(C)}$

2) I & O : $\sqrt{R^2-2Rr}$

Proof :



$AH = 2R\, c_A$

$AO = R$

$\angle HAO = A-(90°-B)-(90°-B)$

$= A+2B-180°$

$= (B-C)$

$$HO = \sqrt{AH^2+AO^2-2AH\cdot AO\cdot\cos(\angle HAO)}$$

$$= \sqrt{4R^2c_A^2+R^2-4R^2c_A\, c_{(B-C)}}$$

$$= R\sqrt{1+4c_A^2-4c_A c_{(B-C)}}$$

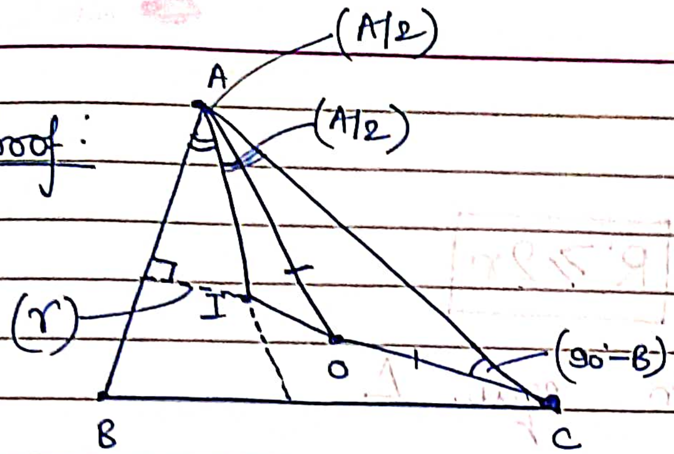
$$= R\sqrt{1+4c_A\left(c_A-c_{(B-C)}\right)}$$

$$= R\sqrt{1+8c_A\, s_{\left(\frac{B-C-A}{2}\right)}\, s_{\left(\frac{A+B-C}{2}\right)}}$$

$$= R\sqrt{1+8c_A\, s_{(B-\pi/2)}\, s_{\left(\frac{\pi}{2}-C\right)}}$$

$$= R\sqrt{1-8c_A c_B c_C}$$

Proof:



$OA = R$

$AI = r / s_{(A/2)}$

$\angle IAO = A/2 - (90° - B)$

$= A/2 + B - \pi/2$

$= (B-C)/2$

$$OI^2 = AI^2 + AO^2 - 2\cdot AI \cdot AO \cdot \cos(\angle IAO)$$

$$= \frac{r^2}{s^2_{(A/2)}} + R^2 - \frac{2Rr}{s_{(A/2)}}\cos\left((B-C)/2\right)$$

$$= R^2 + \frac{r}{s^2_{(A/2)}}\cdot 4R\,\cancel{s_{(A/2)}}\,s_{(B/2)}\,s_{(C/2)} - \frac{2Rr}{s_{(A/2)}}c_{\left(\frac{B-C}{2}\right)}$$

$$= R^2 + \left(\frac{2Rr}{s_{(A/2)}}\right)\left[2s_{\left(\frac{B}{2}\right)}s_{\left(\frac{C}{2}\right)} - c_{\left(\frac{B-C}{2}\right)}\right]$$

$$= R^2 + \left(\frac{2Rr}{s_{(A/2)}}\right)\left(-c_{\left(\frac{B+C}{2}\right)}\right) = \left(R^2 - 2Rr\right)$$

$$\Rightarrow \quad OI = \sqrt{R^2 - 2Rr}$$

Imp. Inequalities

In any Δ, $\boxed{R \geq 2r}$

equality holds for equi. Δ.

Proof: $\cos A + \cos B + \cos C = 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) + 1$

Also, $\cos A + \cos B + \cos C = (2)\left(\cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) + \cos^2\left(\frac{C}{2}\right)\right) - 3$

Now, $\dfrac{\left[\cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) + \cos^2\left(\frac{C}{2}\right)\right]}{3} \leq \cos^2\left(\dfrac{\sum(A/2)}{3}\right) = \cos^2(\pi/6) = \frac{3}{4}$

$\{\cos^2 x$ is Concave Down for $x \in (0, \pi/2)$. $f'' < 0\}$

$\Rightarrow \sum\left(\cos^2(A/2)\right) \leq 9/4 \Rightarrow \left(\sum(\cos A)\right) \leq 3/2$

$\Rightarrow \prod\left(\sin(A/2)\right) \leq 1/8$ $(\leq)$

$\Rightarrow 4\prod\left(\sin A/2\right) \leq 1/2$

$\Rightarrow 4R\left(\prod \sin A/2\right) \leq R/2$

$\Rightarrow r \leq R/2$

$\Rightarrow 2r \leq R$